

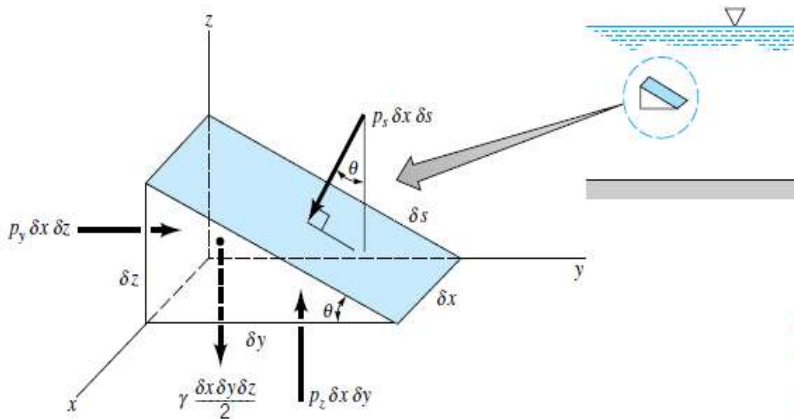
CHAPTER 2: Fluids at Rest- Pressure and its Effects

Introduction

The study of fluids that are at rest or moving in such a manner that there is no relative motion between adjacent particles is what we are going to discuss in this chapter. There will be no shearing stress in the fluid and the only forces that develop on the surfaces of the particles will be due to pressure. Our first concern is to investigate pressure and its variation throughout a fluid and the effect of pressure on submerged surfaces.

2.1 Pressure at a Point

- The term pressure is used to indicate the normal force per unit area at a given point acting on a given plane within the fluid mass of interest. ($P=F/A$)
- **How the pressure at a point varies with the orientation of the plane passing through the point?**
 - First we consider a small triangular wedge of fluid from some arbitrary location within a fluid mass as shown in Fig2.1.
 - The only external forces acting on the wedge are due to the pressure and the weight since we are not considering the shearing stresses ($\tau=0$).
 - Assuming that the shearing stresses will be zero is valid for a fluid element that moves as a rigid body.



■ FIGURE 2.1
Forces on an arbitrary wedge-shaped element of fluid.

- The equations of motion (Newton's Second Law: $F=ma$) in the y and z directions are:
 - Force Balances:
 - $$\sum F_y = p_y \delta x \delta z - p_s \delta x \delta s \sin \theta = \rho \frac{\delta x \delta y \delta z}{2} a_y$$
 - $$\sum F_z = p_z \delta x \delta y - p_s \delta x \delta s \cos \theta - \gamma \frac{\delta x \delta y \delta z}{2} = \rho \frac{\delta x \delta y \delta z}{2} a_z$$

Where: p_s, p_y, p_z = Average Pressures on the faces
 γ = Specific Weight
 ρ = Density
 a_y, a_z = Accelerations

NOTE: A pressure must be multiplied by an appropriate area to obtain the force generated by the pressure.

- $\delta y = \delta s \cos \theta$
- $\delta z = \delta s \sin \theta$

- Therefore the equations of motion can be rewritten as:

$$p_y - p_s = \rho a_y \frac{\delta y}{2}$$

$$p_z - p_s = (\rho a_z + \gamma) \frac{\delta z}{2}$$

- We are interested in what is happening at a point therefore we take the limit as $\delta x, \delta y, \delta z \rightarrow 0$ (while maintaining the angle θ) and it follows that;

$$p_y = p_s \quad p_z = p_s \quad \text{therefore} \quad p_y = p_z = p_s$$

- **Pascal's Law:** The pressure at a point in a fluid at rest, or in motion, is independent of direction as long as there are no shearing stresses present.

2.2 Basic Equation for Pressure Field

- **How does the pressure in a fluid in which there are no shearing stresses vary from point to point?**
- Considering a small rectangular element of fluid removed from some arbitrary position within the mass fluid of interest as shown in the figure below (Fig.2.2).
- There are two type of forces acting on the element:
 - **Surface Forces:** due to pressure
 - **Body Forces:** equal to the weight of the element

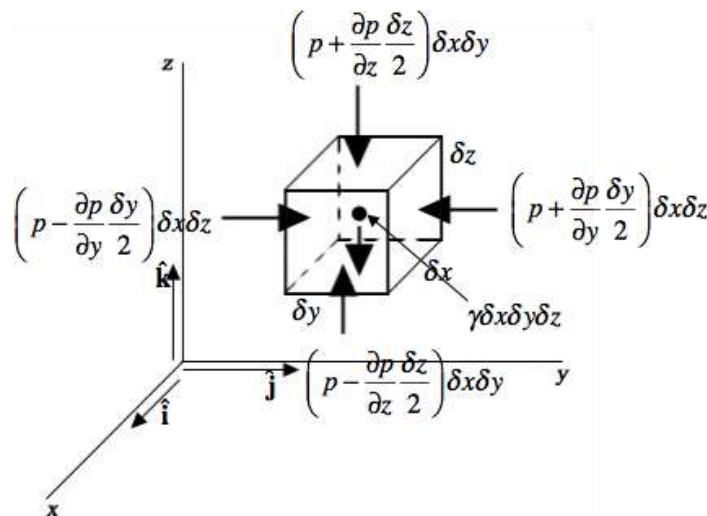


Figure 2.2: Surface and body forces acting on small fluid element

- If we let the pressure at the center of the element be designated as p , then the average pressure on the various faces can be expressed in terms of p and its derivatives as shown in Fig. 2.2.
- We are using a Taylor series expansion at the element center to approximate the pressures a short distance away and neglecting higher order terms that will vanish as we let $\delta x, \delta y$ and δz approach zero. The resultant surface force in the y direction is

$$\delta F_y = \left(p - \frac{\partial p}{\partial y} \frac{\delta y}{2} \right) \delta x \delta z - \left(p + \frac{\partial p}{\partial y} \frac{\delta y}{2} \right) \delta x \delta z$$

or

$$\delta F_y = -\frac{\partial p}{\partial y} \delta x \delta y \delta z$$

- Doing the same for the x and z directions gives us:

$$\delta F_x = -\frac{\partial p}{\partial x} \delta x \delta y \delta z \quad \delta F_z = -\frac{\partial p}{\partial z} \delta x \delta y \delta z$$

- The resultant surface force acting on the element can be expressed in vector form as

$$\delta \vec{F}_s = \delta F_x \hat{i} + \delta F_y \hat{j} + \delta F_z \hat{k}$$

- Substituting the previous results gives us:

$$\delta \vec{F}_s = -\left(\frac{\partial p}{\partial x} \hat{i} + \frac{\partial p}{\partial y} \hat{j} + \frac{\partial p}{\partial z} \hat{k} \right) \delta x \delta y \delta z$$

Where \hat{i} , \hat{j} , and \hat{k} are the unit vectors along the coordinate axes shown in Fig. 2.2.

- We recognize that the group of terms in parentheses represents the vector form of the *pressure gradient* and by employing the del operator, it can be written as

$$\nabla p = \frac{\partial p}{\partial x} \hat{i} + \frac{\partial p}{\partial y} \hat{j} + \frac{\partial p}{\partial z} \hat{k}$$

$$\nabla() = \frac{\partial()}{\partial x} \hat{i} + \frac{\partial()}{\partial y} \hat{j} + \frac{\partial()}{\partial z} \hat{k}$$

- The resultant surface force per unit volume can be expressed as

$$\frac{\delta \vec{F}_s}{\delta x \delta y \delta z} = -\nabla p$$

- Since the z axis is vertical, the weight of the element is

$$-\delta W \hat{\mathbf{k}} = -\gamma \delta x \delta y \delta z \hat{\mathbf{k}}$$

where the negative sign indicates that the force due to the weight is downward and in the negative z direction.

- Newton's second law to be used on a small element can be expressed as:

$$\sum \delta \vec{\mathbf{F}} = \delta m \vec{\mathbf{a}}$$

where $\sum \delta \vec{\mathbf{F}}$ is the resultant force acting on the element, $\vec{\mathbf{a}}$ is the acceleration of the element, and δm is the element mass, which can also be written as $\rho \delta x \delta y \delta z$. This gives us

$$\sum \delta \vec{\mathbf{F}} = \delta \vec{\mathbf{F}}_s - \delta W \hat{\mathbf{k}} = m \vec{\mathbf{a}}$$

or

$$-\nabla p \delta x \delta y \delta z - \gamma \delta x \delta y \delta z \hat{\mathbf{k}} = \rho \delta x \delta y \delta z \vec{\mathbf{a}}$$

simplifying to have

$$-\nabla p - \gamma \hat{\mathbf{k}} = \rho \vec{\mathbf{a}}$$

- This is our general equation of motion for a fluid in where there are no shearing stresses.

Section 2.3: Pressure Variation in a Fluid at Rest

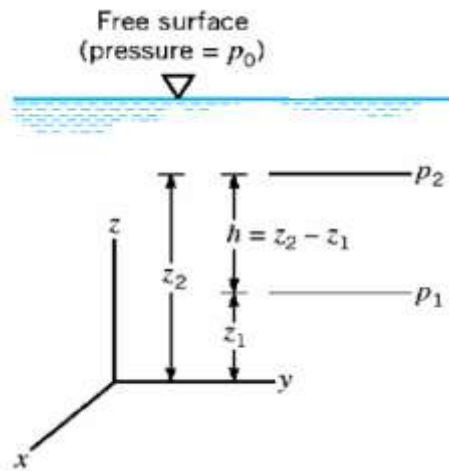
For liquids or gases at rest the pressure gradient in the vertical direction at any point in a fluid depends only on the specific weight of the fluid at that point. The following equation represents this:

$$\frac{\partial p}{\partial z} = -\gamma$$

➤ Incompressible Fluid

An incompressible fluid is a fluid with constant density. For liquids the variation in density is negligible so constant specific weight can be assumed. The equation above is then integrated to

$$\frac{\partial p}{\partial z} = -\gamma \quad \Rightarrow \quad \int_{p_1}^{p_2} dp = -\gamma \int_{z_1}^{z_2} dz$$



$$p_2 - p_1 = -\gamma(z_2 - z_1)$$

$$p_1 - p_2 = \gamma(z_2 - z_1)$$

$$p_1 - p_2 = \gamma h$$

$$p_1 = \gamma h + p_2$$

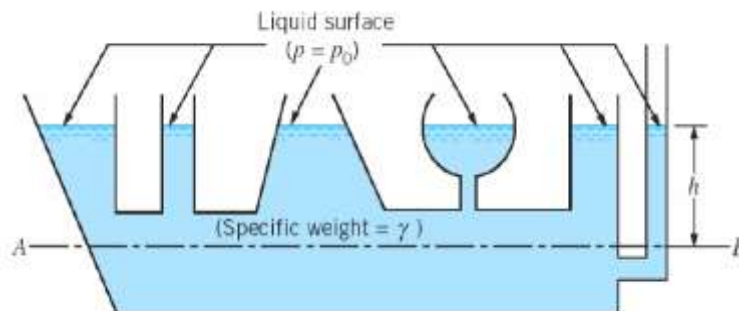
$$h = \frac{p_1 - p_2}{\gamma}$$

The figure above resumes the notation for pressure variation in a fluid at rest with a free surface. As seen above the difference in elevation ($z_2 - z_1$) or the depth of fluid measured downward from the location of p_2 is substituted as h . This pressure distribution is called hydrostatic distribution and shows how the pressure varies linearly with depth. The last equation shown is called pressure head and is interpreted as the height of a column of fluid of specific weight γ required to give a pressure difference ($p_1 - p_2$).

When there is a free surface (as seen in figure), it can be used as a reference plane. The reference pressure p_0 is the pressure acting on the free surface, which will frequently be the atmospheric pressure. Then $p_2 = p_0$ and leads to the following equation:

$$p = \gamma h + p_0$$

The following figure helps us understand how the pressure is the same at all points along the line AB even though the container may have a very irregular shape.



➤ Compressible Fluid

A compressible fluid is one in which the fluid density changes accompanied by changes in pressure and temperature. We mostly think of gases as being compressible; the specific weights of gases are small so the pressure gradient in the vertical direction is small and even over greater distances the pressure remains constant. This way, attention must be given to the variation in specific weight. In order to analyze compressible fluids we use the equation of state for an ideal gas and combine it with the equation for pressure variation in fluids at rest:

$$p = \rho RT \quad \text{and} \quad \frac{\partial p}{\partial z} = -\gamma$$

$$\frac{\partial p}{\partial z} = -\frac{gp}{RT}$$

and by separating variables we get

$$\int_{p_1}^{p_2} \frac{dp}{p} = \ln \frac{p_2}{p_1} = -\frac{g}{R} \int_{z_1}^{z_2} \frac{dz}{T} \quad (2.9)$$

If we assume that the temperature has a constant value:

Isothermal conditions
T = T₀

$$p_2 = p_1 \exp\left[-\frac{g(z_2 - z_1)}{RT_0}\right] \quad (2.10)$$

This equation provides a pressure-variation relationship for an isothermal layer.

2.4 Standard Atmosphere

- Ideally, we would like to have measurements pressure (P) versus altitude (z) over the specific range for the specific conditions (temperature, reference pressure) for which the pressure is to be determined. However, this type of information is usually not available.
- The concept of standard atmosphere was first developed in 1920. The currently accepted standard atmosphere is based on a report published in 1962 and updated in 1976 called **U.S. standard atmosphere**.
- Several important properties for standard atmospheric conditions at sea level are listed in Table 2.1 and figure 2.6 shows the temperature profile for the **U.S. standard atmosphere**.

TABLE 2.1
Properties of U.S. Standard Atmosphere at Sea Level^a

Property	SI Units	BG Units
Temperature, <i>T</i>	288.15 K (15 °C)	518.67 °R (59.00 °F)
Pressure, <i>p</i>	101.33 kPa (abs)	2116.2 lb/ft ² (abs) [14.696 lb/in. ² (abs)]
Density, <i>ρ</i>	1.225 kg/m ³	0.002377 slugs/ft ³
Specific weight, <i>γ</i>	12.014 N/m ³	0.07647 lb/ft ³
Viscosity, <i>μ</i>	1.789 × 10 ⁻⁵ N · s/m ²	3.737 × 10 ⁻⁷ lb · s/ft ²

^aAcceleration of gravity at sea level = 9.807 m/s² = 32.174 ft/s².

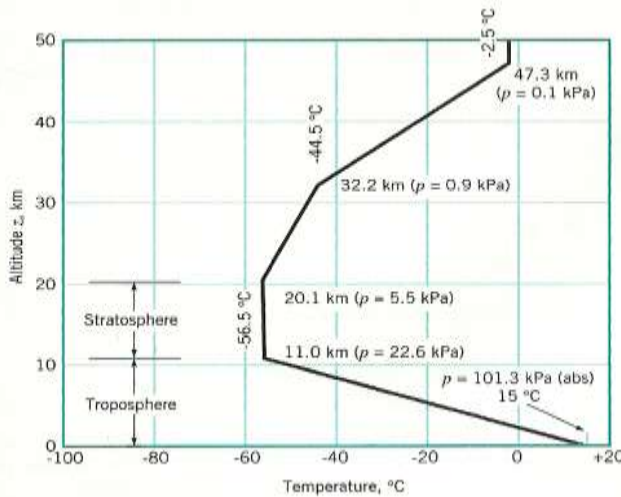


FIGURE 2.6
Variation of temperature with altitude in the U.S. standard atmosphere.

As is shown in this figure, the temperature decreases with altitude in the region nearest the earth's surface (troposphere), then becomes essentially constant in the next layer (stratosphere), and subsequently starts to increase in the next layer.

- Since the temperature variation is represented by a series of linear segments, it is possible to integrate Eq. 2.9 to obtain the corresponding pressure variation.

$$\int_{P_1}^{P_2} \frac{dP}{P} = \ln \frac{P_2}{P_1} = -\frac{g}{R} \int_{z_1}^{z_2} \frac{dz}{T} \quad (2.9)$$

- For example, in the troposphere, which extends to an altitude of about 11 km, the temperature variation is of the form

$$T = T_a - \beta z \quad (2.11)$$

T_a is the temperature at sea level ($z = 0$)

z is the altitude

β is the lapse rate (the rate of change of temperature with elevation). In the troposphere $\beta = 0.00650$

- Equation 2.11 used with Eq. 2.9 yields.

$$P = P_a \left(1 - \frac{\beta z}{T_a}\right)^{g/R\beta} \quad (2.12)$$

P_a is the absolute pressure at $z = 0$

R the gas constant

- P_a , T_a and g are obtained in Table 2.1
- The pressure variation throughout the troposphere can be determined from Eq. 2.12, where the temperature is -56.6°C and absolute pressure is about 23 kPa. Pressures at other altitudes are shown in Fig. 2.6.

2.5 Measurement of Pressure

The pressure at a point; within a fluid mass will be designated as either an *absolute* pressure or a *gage* pressure. Absolute pressure is measured relative to a perfect vacuum (absolute zero pressure), where gage pressure is measured relative to the local atmospheric pressure.

Consequently, a gage pressure of zero corresponds to a pressure that is equal to the local atmospheric pressure.

Absolute pressures are always positive, but gage pressures can be either positive or negative depending on whether the pressure is above atmospheric pressure (a positive value) or below atmospheric pressure (a negative value).

It is important to remember that the standard atmospheric pressure can be expressed as 760 mm Hg or 14.7 psi or 1 atm.

Most likely to measure the atmospheric pressure a mercury barometer is used.

$$p_{\text{atm}} = \gamma h + p_{\text{vapor}} \quad (2.13)$$

where γ is the specific weight of mercury. For most practical purposes the contribution of the vapor pressure can be neglected since it is very small (for mercury, $p_{\text{vapor}} = 0.000023 \text{ lb/in.}^2$ at 68 °F) so it can be considered

$$p_{\text{atm}} \approx \gamma h$$

It is conventional to specify atmospheric pressure in terms of the height, h , in millimeters or inches of mercury.

Note: if water is used instead of mercury, the height of the column would have to be approximately 34 ft rather than 29.9 in. of mercury for an atmospheric pressure of 14.7 psia.

2.6 Manometry

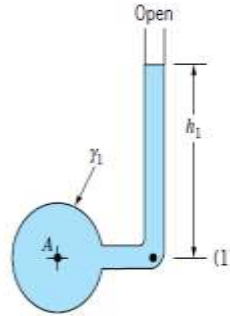
Pressure measuring devices based on this technique are called *manometers*.

Types of manometers:

- Mercury barometer
 - piezometer tube
 - U-tube manometer
 - inclined-tube manometer.
- The **Mercury Barometric** is mostly used to measure atmospheric pressure.

Piezometer Barometer

- Is the simplest type of manometer
- consists of a vertical tube open at the top and attached to the container in which the pressure is desired
- to determine the pressure in the figure 2.9 the following equation can be use:



■ FIGURE 2.9 Piezometer tube.

$$p = \gamma h + p_0$$

through the relationship $p_A = \gamma_1 h_1$

* γ_1 is the specific weight of the liquid in the container.

* because both points have the same elevation. $p_A = p_1$

* h_1 equals the distance between the point 1 and the end of the fluid (see figure 2.9)

Even when the piezometer tube is a very simple and accurate pressure measuring device, has some disadvantages;

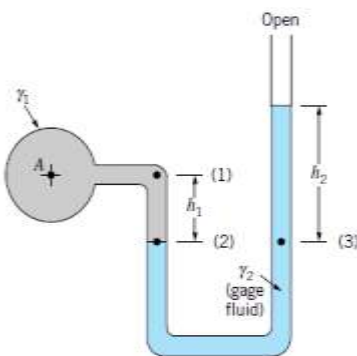
- is only appropriate if the pressure in the container is greater than atmospheric pressure
- the pressure to be measured must be relatively small so the required height of the column is reasonable
- must be a liquid because has an open end.

U-Tube Manometer

To overcome the difficulties of the piezometer tube , the U- tube manometer is used. The fluid in the manometer is called the *gage fluid*. To find the pressure in terms of the various column heights, we start at one end of the system and work our way around to the other end, simply utilizing Eq. 2.8.

$$p = \gamma h + p_0$$

For the U-tube manometer shown in Fig. 2.10



■ FIGURE 2.10 Simple U-tube manometer.

If we start at point A and work around to the open end. The pressure at points A and (1) are the same, and as we move from point (1) to (2) the pressure will increase by $\gamma_1 h_1$ The pressure at point (2) is equal to the

pressure at point (3), since the pressures at equal elevations in a continuous mass of fluid at rest must be the same.

As we move vertically upward the pressure decreases by an amount $\gamma_2 h_2$. In equation form these various steps can be expressed as

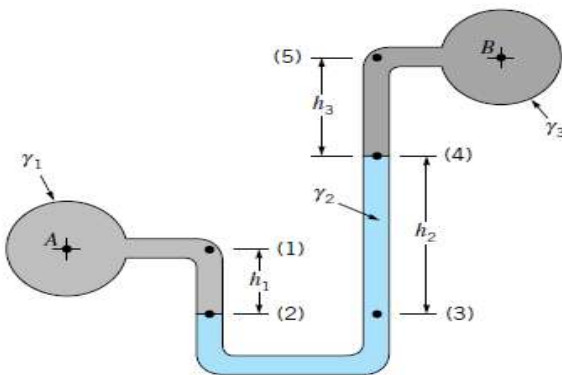
$$p_A + \gamma_1 h_1 - \gamma_2 h_2 = 0$$

therefore, the pressure can be written in terms of the column heights as

$$p_A = \gamma_2 h_2 - \gamma_1 h_1 \quad (2.14)$$

If the fluid A is a gas the equation becomes; $p_A = \gamma_2 h_2$

The U-tube manometer is also widely used to measure the *difference* in pressure between two containers or two points in a given system. Consider a manometer connected between containers A and B as is shown in Fig. 2.11. The difference in pressure between A and B can be found by again starting at one end of the system and working around to the other end.



■ FIGURE 2.11 Differential U-tube manometer.

at A the pressure p_A is which is equal to and as we move to point (2) the pressure increases by $\gamma_1 h_1$. The pressure at p_2 is equal to p_3 , and as we move upward to point (4) the pressure decreases by $\gamma_2 h_2$. Similarly, as we continue to move upward from point (4) to (5) the pressure decreases by $\gamma_3 h_3$. Finally, since they are at equal elevations

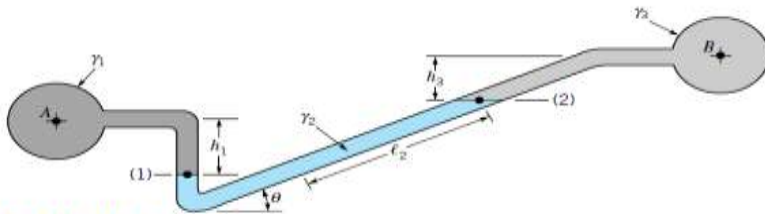
$$p_A + \gamma_1 h_1 - \gamma_2 h_2 - \gamma_3 h_3 = p_B$$

And the difference

$$p_A - p_B = \gamma_2 h_2 + \gamma_3 h_3 - \gamma_1 h_1$$

Inclined-Tube Manometer

To measure small pressure changes, a manometer of the type shown in Fig. 2.12 is frequently used. One leg of the manometer is inclined at an angle θ and the differential reading l_2 is measured along the inclined tube.



■ FIGURE 2.12 Inclined-tube manometer.

The difference in pressure $p_A - p_B$ can be expressed as

$$p_A - p_B = \gamma_2 l_2 \sin \theta + \gamma_3 h_3 - \gamma_1 h_1 \quad (2.15)$$

the pressure difference between points (1) and (2) is due to the *vertical* distance between the points, which can be expressed as $l_2 \sin \theta$. Thus, for relatively small angles the differential reading along the inclined tube can be made large even for small pressure differences. The inclined-tube manometer is often used to measure small differences in gas pressures so that if pipes A and B contain a gas then

$$l_2 = (p_A - p_B) / (\gamma_2 \sin \theta) \quad (2.16)$$

where the contributions of the gas columns h_1 and h_2 have been neglected.

2.7 Mechanical and electronic pressure measuring devices

So far, pressure-measuring techniques involving the use of manometers have been widely discussed. Nonetheless, although the approach is arguably straight-forward and involves simple mathematical manipulations in order to estimate credible pressure measurements, other —more user friendly— devices have been developed.

Essentially, these devices are able to estimate— with varying degree of precision— pressure magnitudes by means of an internal structure deformation that translates to movement of a pointer found on the same instrument.

The **Bourdon pressure gage** is such a device. Because this instrument measures differences between the pressure inside of the deforming instrument and the outside atmospheric pressure, the output value estimated by the instruments' pointer is a gage pressure.

The versatility of this instrument allows measurement of both positive and negative gage values, depending on whether pressures above atmospheric pressures or in vacuum are being estimated, respectively.

The **aneroid barometer** is another mechanical gage used to specifically measure atmospheric pressure. (Bourdon gage is not suitable for this measurements because atmospheric pressure is specified as an absolute pressure).

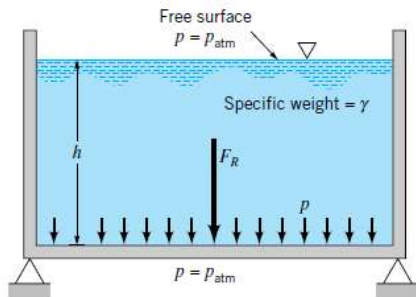
On occasion, we might be interested in measuring pressure changes as a function of time. When such monitoring this is required, devices, a pressure transducer, that are able to convert pressure into electrical outputs are desired.

Such electrical outputs can be digitally stored for future analysis when needed. Implementing this capability into an ordinary Bourdon device can have significant drawbacks on the accuracy of pressure measurements, due to the lack of elasticity of the deforming material found in this instrument. Consequently, a device (elastic diaphragm) with a significantly more elastic material is used.

This elasticity can allow for rapid structural changes in the device and concomitant pressure measurements that are much more accurate than those provided by the Bourdon device.

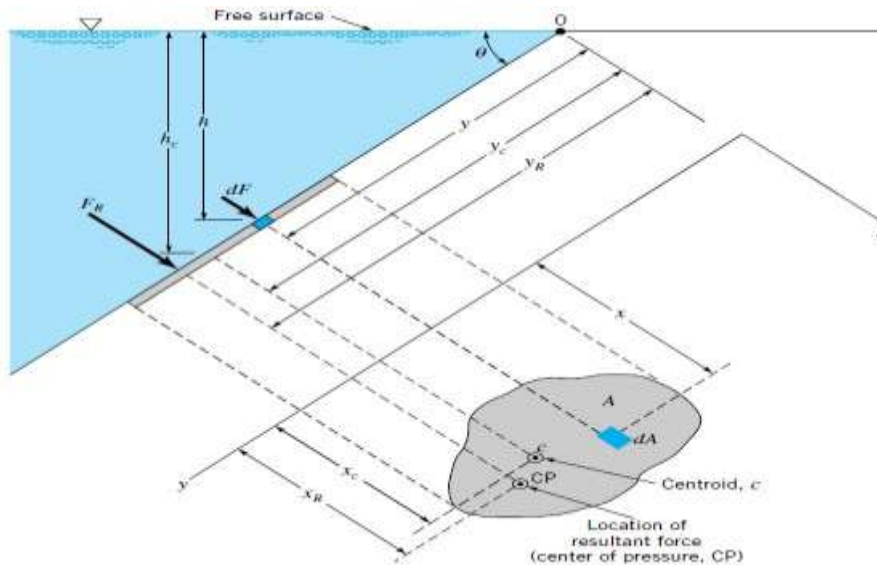
2.8 Hydrostatic Force on a Plane Surface

- When a surface is submerged in a fluid, it experiences forces due to the fluid it is exposed to.
- If the surface is at rest, the force it experiences must be \perp to the surface because there are no shearing stresses.
- If the fluid is incompressible, Pressure will vary linearly with depth.
- Studying these forces is relevant in the design of dams, storage tanks, ships, and other hydraulic structures.
- Analyzing forces in different surfaces:
 - Horizontal Surface; Figure 2.16: Pressure and resultant hydrostatic force developed on the bottom of an open tank



■ **FIGURE 2.16** Pressure and resultant hydrostatic force developed on the bottom of an open tank.

- The magnitude of the resultant force is given by:
 $F_R = PA$ P : uniform pressure on tank's bottom; A : Area of bottom
 $P = \gamma h$ γ : specific weight; h : height
 - When the tank is open, F_R is simply due to the tank's liquid
 - F_R acts through the centroid of the area.
 - Inclined Plane Surface; Figure 2.17: Notation for hydrostatic force on an inclined plane surface of arbitrary shape



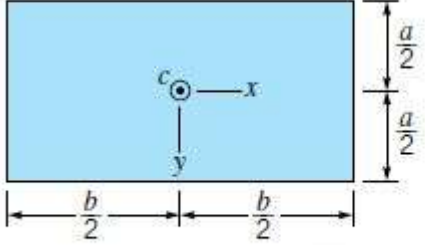
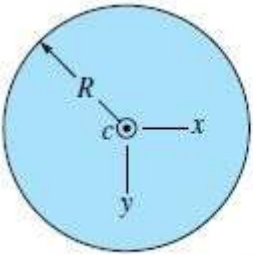
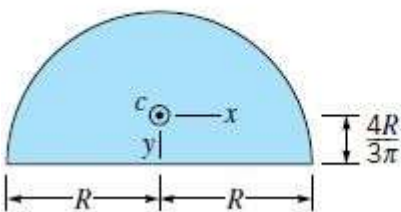
- We assume that the fluid's surface is open to the atmosphere.
- The plane that contains the surface intersects the free surface at O and makes an angle θ with this surface. In the coordinate system O is the origin; y is directed along the surface.
- The area's shape is arbitrary.
- Now, we proceed to find the direction, location, and magnitude of the resultant force acting on one side of the area due to the fluid.
- We have a \perp Force acting on dA at any give depth h given by:
 $dF = \gamma h dA$
- We sum the differential surfaces and find the resultant Force:
 $F_R = \int_A \gamma h dA = \int_A \gamma \sin\theta * y dA$; $h = \sin\theta * y$ γ and θ constant
 $F_R = \gamma \sin\theta \int_A y dA$; *The integral $\int_A y dA$ is called the first moment of Inertia with respect to the x – axis. $\int_A y dA = y_c A$; * y_c represents the y coordinate of the centroid measured from the x-axis which passes through O.
- Going back to the F_R equation and making the substitutions we have:
 $F_R = \gamma A y_c \sin\theta = \gamma h_c A$; h_c : vertical distance from the fluid's surface to the centroid of the area.
- To find the y coordinate of the resultant force we sum moments around x-axis. The moment of the resultant force must equal the moment of the distributed pressure force:
 $F_R y_R = \int_A y dF = \int_A \gamma \sin\theta y^2 dA$; $dF = \gamma h dA \rightarrow h = y \sin\theta$
and $F_R = \gamma A y_c \sin\theta$. Now we substitute for that value and obtain:
 $y_R * y_c A \gamma \sin\theta = \gamma \sin\theta \int_A y^2 dA$. Now we find y_R 's expression:
 $y_R = \frac{\int_A y^2 dA}{A y_c}$; the numerator's integral represents the moment of inertia I_x with respect to an axis formed by the intersection of the plane that contains the surface and the free surface (the x-axis). The expression becomes: $y_R = \frac{I_x}{A y_c}$.
We use the \parallel axes theorem to substitute for I_x in the y_R expression.
 $I_x = I_{xc} + A y_c^2$; I_{xc} : 2nd moment of inertia with respect to an axis passing through its centroid and \parallel to the x-axis.

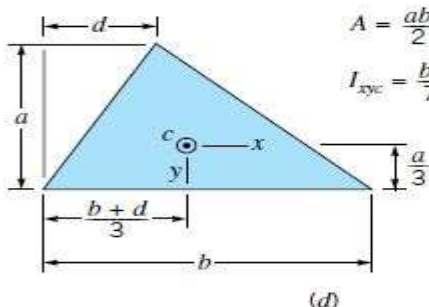
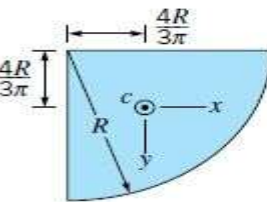
$y_R = \frac{I_{xc}}{AY_c} + y_c$; the resultant force doesn't pass through the centroid. It is always below it. $\frac{I_{xc}}{AY_c} > 0$.

In an analogous way we obtain the x coordinate: $x_R = \frac{\int_A xy dA}{AY_c} = \frac{I_{xy}}{AY_c} = \frac{I_{xyc}}{AY_c} + x$; I_{xy} : product of inertia with respect to the x and y axes, I_{xyc} : product of inertia with respect to an orthogonal coordinate system that passes through the centroid of the area and formed by a translation of the x-y coordinate system.

- Important considerations:
 - As y_c increases, the center of pressure moves closer to the centroid of the area. The 2 ways to increase y_c : (a). increasing the depth of submergence h_c (b). rotating the area for a given depth, decreasing θ .

The following diagram shows the geometric properties of some common shapes:

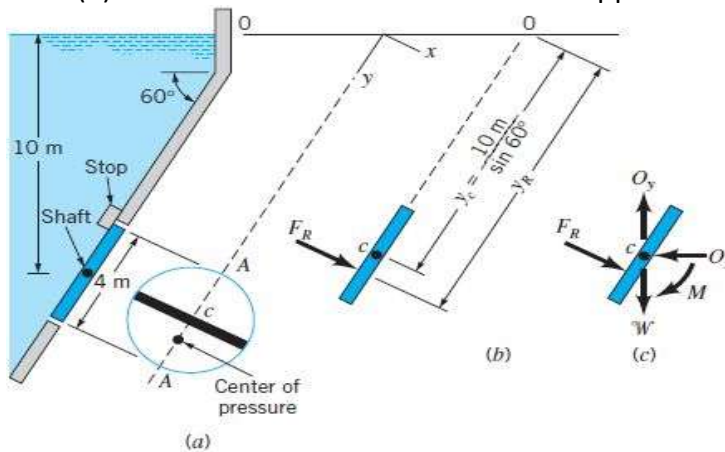
shape	Area and Moments
rectangle	 <div style="display: flex; justify-content: space-between; align-items: flex-start; margin-top: 10px;"> <div style="text-align: center;">(a)</div> <div style="margin-left: 20px;"> $A = ba$ $I_x = \frac{1}{12} ba^3$ $I_y = \frac{1}{12} ab^3$ $I_{xyc} = 0$ </div> </div>
circle	 <div style="display: flex; justify-content: space-between; align-items: flex-start; margin-top: 10px;"> <div style="text-align: center;">(b)</div> <div style="margin-left: 20px;"> $A = \pi R^2$ $I_x = I_y = \frac{\pi R^4}{4}$ $I_{xyc} = 0$ </div> </div>
semicircle	 <div style="display: flex; justify-content: space-between; align-items: flex-start; margin-top: 10px;"> <div style="text-align: center;">(c)</div> <div style="margin-left: 20px;"> $A = \frac{\pi R^2}{2}$ $I_x = 0.1098R^4$ $I_y = 0.3927R^4$ $I_{xyc} = 0$ </div> </div>

triangle	 $A = \frac{ab}{2} \quad I_{xc} = \frac{ba^3}{36}$ $I_{xyc} = \frac{ba^2}{72}(b - 2d)$ <p style="text-align: center;">(d)</p>
One quarter of a circle	 $A = \frac{\pi R^2}{4}$ $I_{xc} = I_{yc} = 0.05488R^4$ $I_{xyc} = -0.01647R^4$ <p style="text-align: center;">(e)</p>

• Examples:

- We have a 4 m diameter circular gate laying on an inclined wall of a large reservoir containing water ($\gamma = 9.80 \times 10^3 \text{ N/m}^3$). The gate is mounted on a shaft along its horizontal diameter. For water depth of 10 m above the shaft determine: (a) the magnitude and location of the resultant force (b) the moment that would have to be applied to open the gate ; Diagram:

Example 1



- (a) 1. Magnitude of force: $F_R = \gamma h_c A = 9.80 \times 10^3 \frac{(N)}{(m^3)} * 10m * \pi 2^2 m^2 = 1.3MN$
 2. Location of force: $x_R = 0$ because of the symmetry of the circular shape; $y_R = \frac{I_{xc}}{AY_c} + y_c$. First we find $I_{xc} = \frac{\pi r^4}{4Y_{cA}}$ and $y_{cA} = \frac{4\pi m^2(2)hc}{\sin\theta}$ where $h_c = 10m$. $y_R = 11.6m$
 → The distance along the gate below the shaft to the center of pressure is $y_R - y_c = 0.0866m$. F_R is \perp to the gate surface.
- (b) Moment required to open the gate: this free body diagram shows that W represents the gate's weight. O_x and O_y are the horizontal and vertical reactions of the shaft on the gate. We sum moments about the shaft:
 $\sum M_c = 0$. $M = F_R (y_R - y_c) = (1230 \times 10^3 N)(0.0866m) = 1.07 \times 10^5 \text{ Nm}$.

Example 2

A fish tank contains seawater ($\gamma = 64.0 \text{ lb/ft}^3$) to a depth of 10ft. A triangular section is replaced with a new one to repair the tank. Determine the magnitude and location of the force the fluid exerts on this area. Diagram:

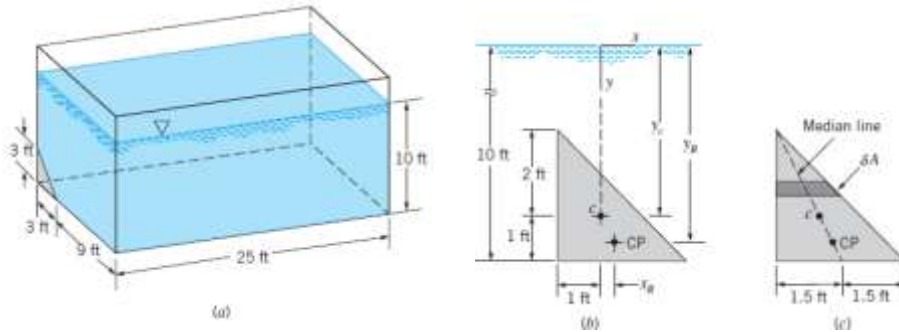


FIGURE E2.7

- (a)1. Magnitude: We obtain the necessary distances for the diagram above. $Y_c = h_c = 9\text{ft}$, because the surface of interest lies in a vertical plane.

The force's magnitude is $F_R = \gamma h_c A = 64.0 \text{ lb/ft}^3 * 9\text{ft} * (9/2\text{ft}^2) = 2590 \text{ lb}$.

*We observe that the length of the tank is irrelevant.

2. Location of the force: $y_R = \frac{I_{xc}}{AY_c} + y$, $I_{xc} = \frac{(3\text{ft})(3\text{ft})^3}{36} = \frac{81 \text{ft}^4}{36}$, then $y_R = \frac{81 * \text{ft}^4 / 36}{9\text{ft} * 9\text{ft}^2 / 2} + 9\text{ft} =$

$0.0556 \text{ ft} + 9\text{ft} = 9.0556 \text{ ft}$. Now we find the x coordinate: $x_R = \frac{I_{xyc}}{AY_c} + x$; $I_{xyc} = \frac{(3\text{ft})(3\text{ft})(3\text{ft})^2}{72} =$

$\frac{81\text{ft}^4}{72}$, finally $x_R = \frac{81\text{ft}^4 / 72}{9\text{ft} * 9\text{ft}^2 / 2} + 0 = 0.0278\text{ft}$

* We see that the coordinates lie on the median line for the area.

2.9 Pressure Prism

- Considering the pressure distribution along a vertical wall of a tank of with b , which contains a liquid having a specific weight γ .
- The pressure must vary linearly with the depth, then we can represent the pressure at the upper surface equal to zero and the pressure at the bottom of the tank equal to γh .
- The figure 2.19 shows the pressure distribution:

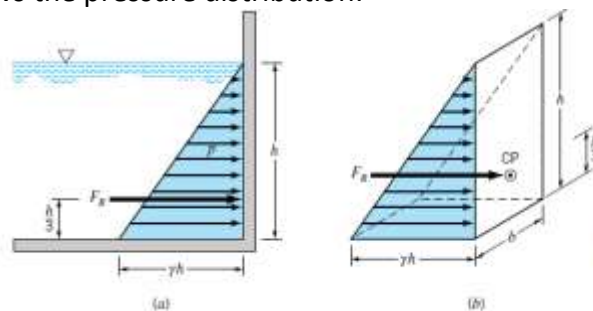


FIGURE 2.19
Pressure prism for vertical rectangular area.

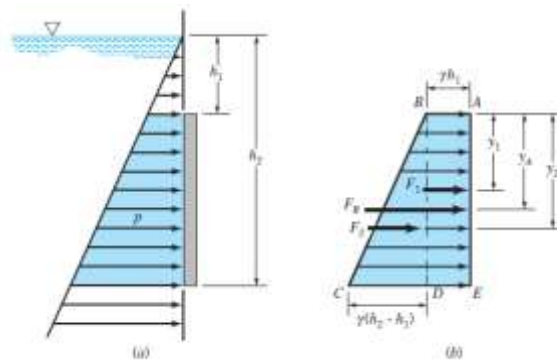
- The average pressure occurs at the depth $h/2$, therefore the resultant force acting on a rectangular area $A=bh$ is:

$$F_R = p_{av}A = \gamma \left(\frac{h}{2} \right) A$$

- The base of this “volume” in the pressure-area space is the plane surface of interest, and its altitude at each point is the pressure.
- This volume is called the **pressure prism**: *the magnitude of the resultant force acting on the surface is equal to the volume of the pressure prism.*

$$F_R = volume = \frac{1}{2}(\gamma h)(bh) = \gamma \left(\frac{h}{2} \right) A$$

- The magnitude of the resultant fluid force is equal to the volume of the pressure prism and passes through its *centroid*.
- The centroid is located along the vertical axis of symmetry of the surface and at a distance of $h/3$ above the base (since the centroid of a triangle is located at $h/3$ above its base).
- The same graphical approach can be used for plane surfaces that do not extend up to the fluid surfaces as shown in Figure 2.20.



■ FIGURE 2.20 Graphical representation of hydrostatic forces on a vertical rectangular surface.

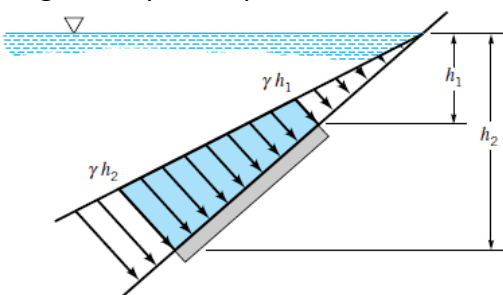
- At this specific case the cross section of the pressure prism is trapezoidal. However the resultant force is still equal in magnitude to the volume of the pressure prism and it passes through the centroid of the volume.
- Pressure prism can be decomposed into two parts, ABDE and BCD, thus

$$F_R = F_1 + F_2$$

- The location of F_R can be determined by summing moments about some convenient axis such as one passing through A. In this instance y_1 & y_2 can be determined by inspection.

$$F_R y_A = F_1 y_1 + F_2 y_2$$

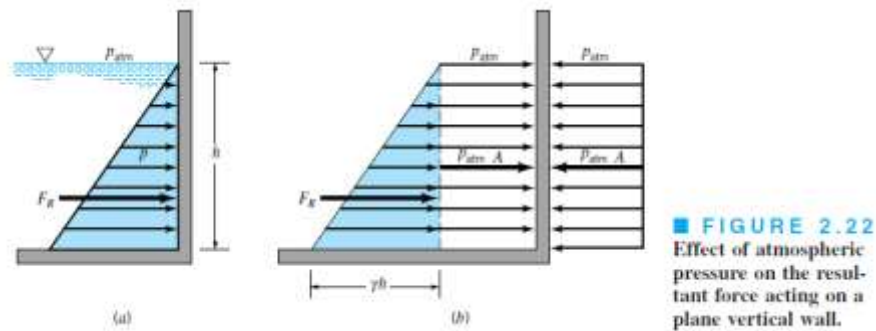
- For inclined plane surfaces the pressure prism can still be developed and the cross section of the prism will generally be trapezoidal as shown in the Fig.2.21.



■ FIGURE 2.21 Pressure variation along an inclined plane area.

- Using pressure prisms for determining the force on submerged plane areas is convenient if the area is rectangular so the volume and centroid can be easily determined.

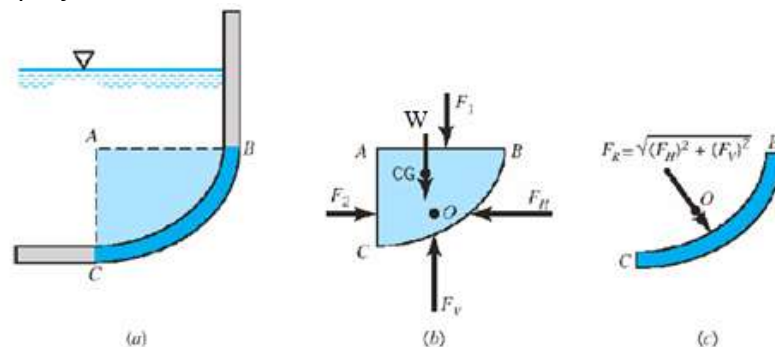
- When we have nonrectangular shapes, integration would generally be needed to determine the volume and centroid.
- **How will atmospheric pressure influence the resultant force?**
- Earlier we were considering atmospheric pressure equal to zero, and thus the pressure used in the determination of the fluid force is gauge pressure.
- Now we want to include atmospheric pressure therefore in this case the force on one side of the wall we must realize that this same pressure acts on the outside surface, so an equal opposite force will be developed.



- We conclude that the resultant fluid force on the surface is that due only to the gage pressure contribution of the liquid in contact with the surface-the atmospheric pressure does not contribute to the resultant force.
- If the surface pressure of the liquid is different from atmospheric pressure the resultant force acting on a submerged area, A , will change in magnitude from that caused simply by hydrostatic pressure.

2.10: Hydrostatic Force on a Curved Surface

The equations developed in earlier sections for forces on a submerged surface only apply to plane surfaces. As we know many surfaces of interest are nonplanar. In order to measure the resultant force, we will consider the equilibrium of the fluid volume enclosed by a curved surface of interest and the horizontal and vertical projections of the surface.



As we can see in the figure above, the development of a free-body diagram of the volume of the fluid is used to determine the resultant force acting on the curved surface. Forces F_1 and F_2 can be determined from the relationships for planar surfaces. The weight W , acts through the center of gravity CG . The forces F_H and F_V represent the components of the force that the tank exerts on the fluid. In order for the force system to be in equilibrium:

$$F_H = F_2 \quad F_V = F_1 + W$$

The magnitude of the resultant force can be obtained with the following equation:

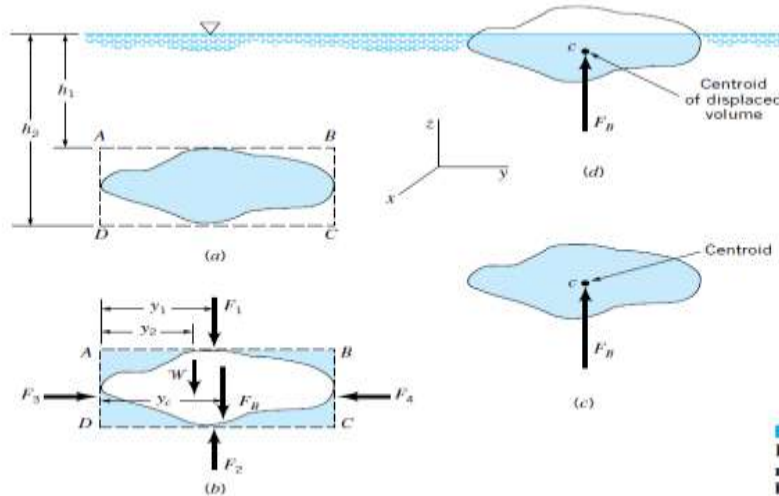
$$F_R = \sqrt{(F_H)^2 + (F_V)^2}$$

The resultant force F_R passes through the point O , which can be located by summing moments about an appropriate axis.

2.11 Buoyancy, Flotation and Stability

- **Buoyant force:** A resultant body force that is generated when a stationary body is completely or partially submerged in a fluid.
- **Archimedes' principle:** A body wholly or partly immersed in a fluid is buoyed up by a force equal to the weight of the fluid it displaces. The buoyant force can be considered to act vertically upward through the center of gravity of the displaced fluid.

$F_B = \text{buoyant force} = \text{weight of displaced fluid}$



■ FIGURE 2.24
Buoyant force on submerged and floating bodies.

The location of the line of action of the buoyant force can be determined by summing moments of the forces shown on the free-body diagram in Fig. 2.24b with respect to some convenient axis. For example, summing moments about an axis perpendicular to the paper through point D we have

$$\begin{aligned} F_B y_c &= F_2 y_1 - F_1 y_1 - W y_2 && \text{after some substitutions;} \\ \mathcal{V} y_c &= \mathcal{V}_T y_1 - (\mathcal{V}_T - \mathcal{V}) y_2 && (2.23) \end{aligned}$$

\mathcal{V}_T is the total volume $(h_2 - h_1)A$. The right-hand side of Eq. 2.23 is the first moment of the displaced volume \mathcal{V} with respect to the x - z plane so that y_c is equal to the y coordinate of the centroid of the volume \mathcal{V} . In a similar fashion it can be shown that the x coordinate of the buoyant force coincides with the x coordinate of the

centroid. Thus, we conclude that the *buoyant force passes through the centroid of the displaced volume* as shown in Fig. 2.24c. The point through which the buoyant force acts is called the *center of buoyancy*.

These same results apply to floating bodies which are only partially submerged, as illustrated in Fig. 2.24d, if the specific weight of the fluid above the liquid surface is very small compared with the liquid in which the body floats. Since the fluid above the surface is usually air, for practical purposes this condition is satisfied.

Stability

An additional factor to be considered when dealing with floating or submerged bodies is their stability when external factors interact with them. These interactions can displace the body's equilibrium position. If such displacement is **stable** and the body is completely submerged in a fluid, a restoring couple between the weight and buoyant force will be formed and, consequently, the body will return to its original equilibrium position. If **unstable**, the body will move to a new equilibrium position once it is displaced. Factors that determine whether a restoring couple or overturning couple will be induced.

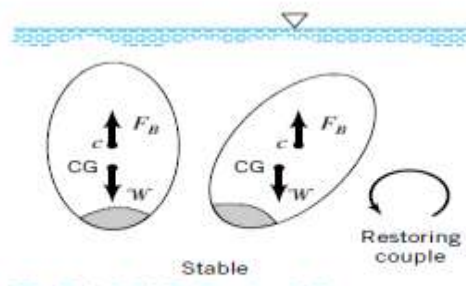


FIGURE 2.25
Stability of a completely immersed body — center of gravity below centroid.

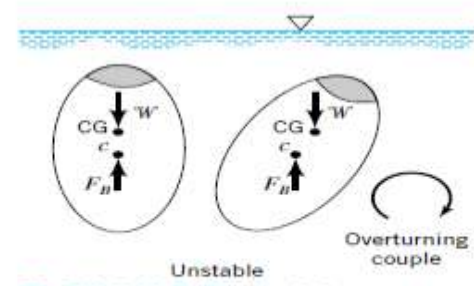


FIGURE 2.26
Stability of a completely immersed body — center of gravity above centroid.

- **Restoring couple** =the body's center of gravity is below its centroid.
- **Overturning couple** =the body's center of gravity is found above its centroid.

In the case of floating bodies the analysis gains intricacy, for geometrical and weight distribution factors need to be considered when determining whether a restoring or overturning couple forms once the floating body's position is displaced.

Essentially, when floating bodies are displaced from their equilibrium position, a concomitant shift in position of the buoyant force is also observed. If this shift can couple with the body's weight in such a way that the body is not further displaced but returned to its original position, we have then encountered a restoring couple.

Consider as an example a lever (shown below), where two blocks (I and II) of equal weight are initially placed. If a third block (shown in red) is placed on top of block II, we will encounter a displacement of the lever from equilibrium position. Nonetheless, if the weight of this third block is such that it does not contribute significantly to the overall weight of block II, the lever will eventually return to equilibrium position, analogous to the restoring couple described for floating bodies.

Although this is a poor example that does not apply considerably to our present study of floating bodies, it sheds some light on a concept that might be initially puzzling.

2.12 Pressure Variation in a Fluid with Rigid-Body Motion.

General equation of motion. Eq. 2.2

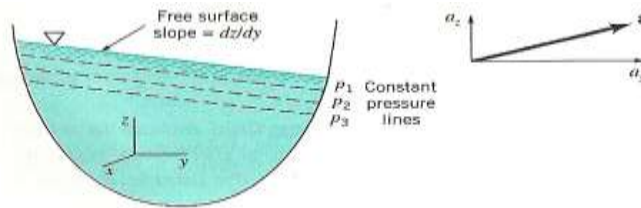
$$-\nabla p - \gamma \hat{k} = \rho \mathbf{a}$$

- Equation 2.2 based on rectangular coordinates with the positive z axis being vertically upward, can be expressed as

$$-\frac{\partial p}{\partial x} = \rho a_x \quad -\frac{\partial p}{\partial y} = \rho a_y \quad -\frac{\partial p}{\partial z} = \gamma + \rho a_z \quad (2.24)$$

Linear motion

- We first consider an open container of a liquid that is translating along a straight path with a constant acceleration \mathbf{a} as illustrated in Fig. 2.29. Since $a_x = 0$, the pressure gradient in the x direction is zero ($\partial p / \partial x = 0$). In the y and z direction



■ FIGURE 2.29
Linear acceleration of a liquid with a free surface.

- In the y and z direction:

$$\frac{\partial p}{\partial y} = -\rho a_y \quad \frac{\partial p}{\partial z} = -\rho(g + a_y) \quad (2.25, 2.26)$$

- The change pressure between two closed spaced can be expressed as

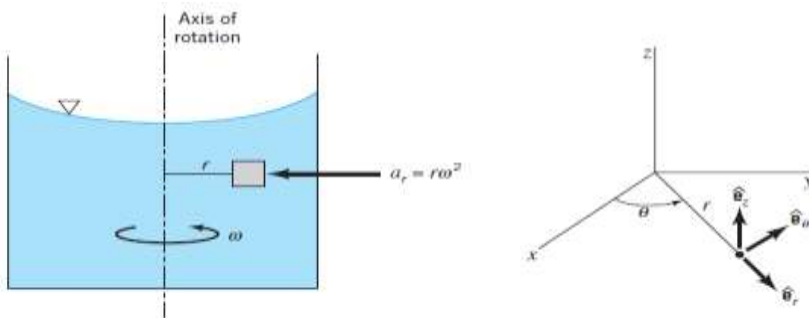
$$dp = \frac{dp}{dy} dy + \frac{dp}{dz} dz = -\rho a_y + -\rho(g + a_y) \quad (2.27)$$

- Along a line of constant pressure $dp = 0$ therefore the slope of the line is given by the relationship

$$\frac{dz}{dy} = -\frac{a_y}{g + a_y} \quad (2.28)$$

Rigid-Body Rotation

- A fluid contained in a tank that rotates with a constant angular velocity ω about an axis as is shown in Fig. 2.30 will rotate with the tank as a rigid body.



■ FIGURE 2.30
Rigid-body rotation of a liquid in a tank.

- In term of cylindrical coordinates the pressure gradient can be expressed

$$\nabla p = \frac{\partial p}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial p}{\partial \theta} \hat{e}_\theta + \frac{\partial p}{\partial z} \hat{e}_z \quad (2.29)$$

- It is known from elementary particle dynamics that the acceleration of a fluid particle located at a distance r from the axis of rotation is equal in magnitude to $r\omega^2$, and the direction of the acceleration is toward the axis of rotation.

- In terms of this coordinate system

$$a_r = -r\omega^2 \hat{e}_r \quad a_\theta = 0 \quad a_z = 0$$

- and from equation 2.2

$$\frac{\partial p}{\partial r} = \rho r \omega^2 \quad \frac{\partial p}{\partial \theta} = 0 \quad \frac{\partial p}{\partial z} = -\gamma \quad (2.30)$$

- The differential pressure is

$$dp = \frac{dp}{dr} dr + \frac{dp}{dz} dz = \rho r \omega^2 dr - \gamma dz \quad (2.31)$$

- Along surface of constant pressure, $dp = 0$, so that from Eq. 2.31 (using $\gamma = \rho g$)

- $\frac{dz}{dr} = \frac{r\omega^2}{g}$

- Therefore the equation for surfaces of constant pressure is

- $z = \frac{\omega^2 r^2}{2g} + \text{constant} \quad (2.32)$

- This equation reveals that these surfaces of constant pressure are parabolic as illustrated in Fig 2.31

$$\int dp = \rho \omega^2 \int r dr - \gamma \int dz$$

or

$$p = \frac{\rho \omega^2 r^2}{2} - \gamma z + \text{constant} \quad (2.33)$$

References

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Class Notes Inqu 4010 Professor Cordova Ph.D. January 28, 2010, February 2, 2010, February 4, 2010

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Maria Canales, Amalchi Castillo, Darlene Colon, Karen Collazo, Angelica Cortes and Irmarié Cotto