Chapter 7: **Dimensional Analysis, Modeling and Similitude**.

The solution to many engineering problems is achieved through the use of a combination of analysis and experimental data. One of the goals of an experiment is to make the results as widely applicable as possible. That's why the concept of similitude is often used so that measurements made in the laboratory, for example, can be used to describe the behavior of other systems outside the laboratory by establishing the relationship between the laboratory model and the outside system. The following sections explain how this can be done in a systematic manner.

Section 7.1: Dimensional Analysis

The steady flow of an incompressible Newtonian fluid through a long, smooth- walled, horizontal, circular pipe can be an example of a typical fluid mechanics problem in which experimentation is required. The pressure drop per unit length that develops along the pipe as a result of friction is an important characteristic of the system. The calculation of it may seem very simple but it cannot be solved analytically without the use of experimental data.

The first step for conducting such experiment would be to decide on the variables that will affect the pressure drop per unit length. We must take to consideration the pipe diameter D, the fluid density ρ, fluid viscosity μ, and a mean velocity V, at which the fluid is flowing through the pipe. This relationship is expressed as:

$$\Delta\rho_l = f\,(D,\rho,\mu,V) \qquad \mathbf{(7.1)}$$

This simply indicates mathematically that we expect the pressure drop per unit length to be some function of the factors contained within the parentheses. Since the nature of the function is unknown the experiments' objective is to determine the nature of this function.

It is important to develop a meaningful and systematic way to perform an experiment.

Once you examine carefully how to perform the experiment you realize that it must be done leaving three variables constant while one of them is changing, obtaining a series of plots done analytically (see figure 7.1). But, how can you vary fluid density while holding viscosity constant at the laboratory? If there's a way of obtaining experimental data that resemble the analytical plots, how could you combine these data to obtain a general functional relationship between $\Delta\rho_l$, D,ρ,μ and V which would be valid for any similar pipe system?

Luckily there's a much simpler approach to this problem. Instead of working with the original list of variables in Eq. 7.1, you can collect these into two nondimensional combinations of variables called dimensionless products or dimensionless groups, so that

(7.2)

$$\frac{D\Delta pl}{\rho V^2}=\frac{\rho VD}{\mu}$$

Now you only have to work with two variables instead of five. The experimental part would only consist of varying the dimensionless product $\rho VD/\mu$ and determining the corresponding value of $D\Delta p_l/\rho V^2$. The results can be represented in a single curve as shown in Figure 7.2. This curve would be valid for any combination of smooth-walled pipe and incompressible Newtonian fluid. To obtain such curve you can choose a pipe of convenient size and a fluid that's easy to work with. It is not necessary to use different pipes or different fluids. This makes the experiment very simple and less expensive.

From Chapter 1 recall the terms of basic dimensions like mass, M, length, L, and time, T. Also you can use force, F, L, and T as basic dimensions, since from Newton's second law

$$F=MLT^{-2}$$

The dimensions of the variables in the pipe flow example are

$\Delta p_l=FL^{-3}$ $D=L$ $\rho=FL^{-4}T^2$ $\mu=FL^{-2}T$ $V=LT^{-1}$

Note that the pressure drop per unit length has the dimensions of $(F/L^2)/L=FL^{-3}$.

When the dimensions of the two groups in Eq. 7.2 are checked you come to realize that they are dimensionless products.

$$\frac{D\Delta pl}{\rho V^2}=\frac{L\left(\frac{F}{L^3}\right)}{(FL^{-4}T^2)\left(\frac{L}{T}\right)^2}=F^0L^0T^0$$

And

$$\frac{\rho VD}{\mu}=\frac{(FL^{-4}T^2)(LT^{-1})(L)}{(FL^{-2}T)}=F^0L^0T^0$$

Not only you reduced from five to two variables, but the new groups will be independent of the system units you choose to use. This type of analysis is called dimensional analysis and it will be used in the next section.

7.2 Buckingham Pi Theorem

If an equation involving k variables is dimensionally homogeneous, it can be reduced to a relationship among k - r independent dimensionless products, where r is the minimum number of reference dimensions required to describe the variables.

The dimensionless products are frequently referred to as "pi terms", and the theorem is called the Buckingham pi theorem. To represent a dimensionless product the π symbol is used. The pi theorem is based on the idea of dimensional homogeneity which was introduced in Chapter 1. Essentially we assume that for any physically meaningful equation involving k variables, such as

$u_1 = f(u_2, u_3, \ldots, u_k)$

the dimensions of the left side of the equal sign must be equal to the dimensions of any term that stands by itself on the right side of the equal sign. It then follows that you can arrange the equation into a set of dimensionless products (pi terms) so that

$\pi_1 = \phi(\pi_2, \pi_3, \ldots, \pi_{k-r})$

where $\phi(\pi_2, \pi_3, \ldots, \pi_{k-r})$ is a function of π_2 through π_{k-r}.

The required number of pi terms is fewer than the number of original variables by r, where r is determined by the minimum number of reference dimensions required to describe the original list of variables. Usually the reference dimensions required to describe the variables will be the basic dimensions M, L, and T or F, L, and T. However, in some instances perhaps only two dimensions, such as L and T, are required or maybe just one, such as L. Also, in a few rare cases the variables may be described by some combination of basic dimensions, such as M/T^2, and L, and in this case r would be equal to two rather than three. With the pi theorem you can develop a simple, systematic procedure for developing the pi terms for a given problem.

7.3 Determination of Pi Terms

The method of repeating variables is a dimensional analysis that is performed by using a series of distinct steps.

Step 1 List all the variables that are involved in the problem

Be sure to include all the variables that affect the system in this step, otherwise the whole process will be incorrect. But, do not overdo it since you have to study these independent variables experimentally.

Step 2 Express each of the variables in terms of basic dimensions

Check Table 1.1 in Chapter 1.

Step 3 Determine the required number of pi terms

From the Buckingham pi theorem the number of pi terms is equal to k-r. k is obtained in step one and r is obtained in step two. See example 7.2.

Step 4 Select a number of repeating variables, where the number required is equal the number of reference dimensions

Select variables, from the original list, that you can combine with each of the remaining variables to form a pi term. Repeating variables cannot themselves be combined to form a dimensionless product.

Step 5 Form a pi term by multiplying one of the no repeating variables by the product of the repeating variables, each raised to an exponent that will make the combination dimensionless

Each pi term will be of the form $u_i u_1^{a_i} u_2^{b_i} u_3^{c_i}$ where u_i is one of the non repeating variables; u_1, u_2, and u_3 are the repeating variables; and the exponents a_i, b_i, and c_i are determined so that the combination is dimensionless.

Step 6 Repeat Step 5 for each of the non repeating variables

The resulting set of pi terms will correspond to the required number obtained from Step 3. Check for errors otherwise.

Step 7 Check all the resulting pi terms to make sure they are dimensionless

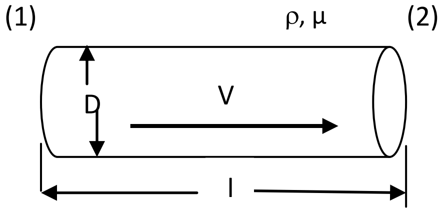
Substitute the dimensions of the variables into the pi terms to confirm that they are all dimensionless.

Step 8 Express the final form as a relationship among the pi terms, and think about what it means

The final form can be written as $\pi_1 = \phi(\pi_2, \pi_3, \ldots, \pi_{k-r})$, where π_1 would contain the dependent variable in the numerator. If everything is correct the relationship of π_1 in terms of the pi terms can be used to describe the problem. You need only work with the pi terms- not the individual variables.

Remember the problem discussed earlier which was concerned with the steady flow of an incompressible Newtonian fluid through a long, smooth- walled, horizontal circular pipe? Now we'll solve it in detail with the method of repeating variables.

You are interested in the pressure drop per unit length, Δp_l, along the pipe as illustrated by the figure.



$\Delta p_l = (p_1 - p_2)/l$

$\Delta p_l = f(D, \rho, \mu, V)$

In Step 2 you express all the variables in terms of basic dimensions.

In Step 3 you apply the pi theorem to determine the required number of pi terms.

You should give special attention to the selection of repeating variables as detailed in Step 4.

In this step you have to choose the repeating variables to be used to form the pi terms from the list D,ρ,μ, and V.

Step 5 is the formation of the two pi terms. For more details see page 352-353.

In Step 6 the process is repeated for the remaining non repeating variables.

In step 7 check using both FLT and MLT dimensions.

In step 8 express the result of the dimensional analysis.

Π_2= ρVD/ μ,

and The relationship between Π_1 and Π_2 is $\frac{D\Delta p_e}{\rho V^2} = \emptyset\ \frac{\rho VD}{\mu}$. The dimensionless product $\frac{\rho VD}{\mu}$ is the Reynolds number. A summarize of the steps to be followed in performing a dimensional analysis using the method of repeating variables are as follows:
Step 1: List all the variables that are involved in the problem.
Step 2: Express each of the variables in terms of basic dimensions.
Step 3: Determine the required number of pi terms.
Step 4: Select a number of repeating variables, where the number required is equal to the number of reference dimensions.
Step 5: Form a pi term by multiplying one of the non-repeating variables by the product of repeating variables each raised to an exponent that will make the combination dimensionless.
Step 6: Repeat the step 5 for each of the remaining non-repeating variables.
Step 7: Check all the resulting pi terms to make sure they are dimensionless.
Step 8: Express the final form as a relationship among the pi terms and think about what it means.

7.4 Some Additional Comments about Dimensional Analysis

Other methods could be used to performing a dimensional analysis, pi terms can also be formed by inspection. Regardless of the specific method used for the dimensional analysis, there are certain aspects of this important engineering tool that must seem a little baffling and mysterious to the student.

7.4.1 Selection of Variables

The term variable is used to indicate any quantity involved, including dimensional and non-dimensional constants. It is imperative that sufficient time and attention be given to this first step in which the variables are determined. Most engineering problems involve certain simplifying assumptions that have an influence on the variables to be considered. It is often helpful to classify variables into three groups: geometry, material properties, and external effects.

- **Geometry:** The geometric characteristics can usually be described by a series of lengths and angles. In most problems the geometry of the system plays an important role, and a sufficient number of geometric variables must be included to describe the system. These variables can usually be readily identified.

- **Material Properties:** Since the response of a system to applied external effects such as force, pressure, and changes in temperature is dependent on the nature of the materials involved in a system, the material properties that relate the external effects and the responses must be included as variables.
- **External Effects:** This terminology is used to denote any variable that produces, or tends to produce a change in the system. For example, in structural mechanics, force applied to a system tends to change its geometry and such forces would need to be considered as pertinent variables. For fluid mechanics, variables in this class would be related to pressure, velocities, or gravity.

If we have problem in which the variables are $f(p,q,r,\ldots.,u,v,w,\ldots)=0$ and it is known that there is an additional relationship among some of the variables, for example, $f_1(u,v,w,\ldots)$ the q is not required and can be omitted. Conversely, if it is known that the only way the variables u,v,w…. enter the problem is through the relationship expressed equation, then the variables u,v,w,… can replaced by the single variable q, therefore reducing the number of variables. In summary, the following points should be considered in the selection of variables:

1. Clearly define the problem. Define the main variable of interest (the dependent variable)?
2. Consider the basic laws that govern the phenomenon. Even a crude theory that describes the essential aspects of the system may be helpful.
3. Start the variable selection process by grouping the variables into three broad classes: geometry, material properties, and external effects.
4. Consider other variables that may not fall into one of the above categories. For example, time will be an important variable if any of the variables are time dependent.
5. Be sure to include all quantities that enter the problem even though some of them may be held constant. For a dimensional analysis it is the dimensions of the quantities that are important.
6. Make sure that all variables are independent. Look for relationship among subsets of the variables.

7.4.2 Determination of Reference Dimensions

For any given problem it is obviously desirable to reduce the number of pi terms to a minimum and, therefore we wish to reduce the number of variables to a minimum, we certainly do not want to include extraneous variables. The use of FLT or MLT as basic dimensions is the simplest, and these dimensions can be used to describe fluid mechanical phenomena. Typically, in fluid mechanics, the required number of reference dimensions is three, but in some problems only one or two are required.

7.4.3 Uniqueness of Pi Terms

In the problem of studying the pressure in a pipe, we selected D,V and ρ as repeating variables. This led to the formulation of the problem in terms of pi terms as

$$\frac{D\Delta p_e}{\rho V^2} = \emptyset\ \frac{\rho VD}{\mu}$$

If we had selected D, V and μ as repeating variables the pi term involving Δp_e becomes

$$\frac{\Delta p_e D^2}{V\mu}$$

and the second pi terms remains the same. Thus we can express the final result as

$$\frac{\Delta p_e D^2}{V\mu} = \emptyset_1 \frac{\rho V D}{\mu}$$

Both results are correct, and both would lead to the same final equation for Δp_e. The functions $\emptyset$ and $\emptyset_1$will be different because the dependent pi terms are different for the two relationships. We can conclude that there is not a unique set of pi terms which arises from a dimensional analysis. Once a correct set of pi terms is obtained, any other set can be obtained by manipulation of the original set. If we have a problem involving, say, three pi terms, $\Pi_1 = \phi(\Pi_2, \Pi_3)$ we could always form a new set from this one by combining the pi terms. For example, we could form a new pi term, $\Pi_{2,}^{'}$ by letting $\Pi_2^{'} = \Pi_2^a \Pi_3^b$ where a and b are arbitrary exponents. Then the relationship could be expressed as $\Pi_1 = \emptyset_1(\Pi_2^{'}, \Pi_3)$ or $\Pi_1 = \emptyset_2 (\Pi_2 , \Pi_2^{'},)$. There is no unique set of pi terms for a given problem; the number required is fixed in accordance with the pi theorem.

7.5 Determination of Pi Terms by Inspection

The method of repeating variables for forming pi terms is simple and straightforward, it is rather tedious, particularly for problems in which large numbers of variables are involved. Since the only restrictions placed on the pi terms are that they be (1) correct in number, (2) dimensionless, and (3) independent, it is possible to simply form the pi terms by inspection, without resorting to the more formal procedure. To illustrate this approach, we again consider the pressure drop per unit length along a smooth pipe. The starting point remains the same determine the variables, which in this case are

$$\Delta p_e = f(D, \rho, \mu, V)$$

Next, the dimensions of the variables are listed:

$$\Delta p_e = FL^{-3}$$
$$D = L$$
$$\rho = FL^{-4}T^2$$
$$\mu = FL^{-2}T$$
$$V = LT^{-1}$$

Then the number of reference dimensions determined. The application of the pi theorem tells us how many pi terms are required. In this example, since there are five variables and three reference dimensions, two pi terms are needed. Pi terms can be formed by inspection by simply making use of the fact that each pi term must be dimensionless. We will always let Π_1 contain the dependent variable, which in this example is Δp_e. Because this variable has the dimensions FL^{-3}, we need to combine it with other variables so that a non-dimensional product will result. One possibility is to first divide Δp_e by ρ (to eliminate F), then divide by V^2 (to eliminate T), and finally multiply by D to make the combination dimensionless.

(1) $$\frac{\Delta p_e}{\rho} = \frac{(FL^{-3})}{(FL^{-4}T^2)} = \frac{L}{T^2}$$

(2) $$\left(\frac{\Delta p_e}{\rho}\right)\frac{1}{V^2} = \left(\frac{L}{T^2}\right)\frac{1}{(LT^{-1})^2} = \frac{1}{L}$$

(3) $$\left(\frac{\Delta p_e}{\rho V^2}\right) D = \left(\frac{L}{L}\right)(L) = L^0$$

$$\Pi_1 = \frac{\Delta p_e D}{\rho V^2}$$

Next we will form the second pi term by selecting the variable that was not used in Π_1, in this case is μ.

$$\prod_2 = \frac{\mu}{\rho V D} = \frac{(FL^{-2}T)}{(FL^{-1}T^2)(LT^{-1})(L)} = F^0 L^0 T^0$$

$$\frac{\Delta p_t D}{\rho V^2} = \emptyset\left(\frac{\mu}{\rho V D}\right)$$

The $\prod_2$ can be formed by the combination of say $\prod_3, \prod_4$ *and* $\prod_5$ such as:

$$\prod_2 = \frac{\prod_3^2 \prod_4}{\prod_5}$$

then $\prod_2$ is not an independent pi term.

7.6 Common Dimensionless Group in Fluid Mechanics

The **Reynolds number** is undoubtedly the most famous dimensionless parameter in fluid mechanics. It is named in honor of Osborne Reynolds (1842–1912), a British engineer who first demonstrated that this combination of variables could be used as a criterion to distinguish between laminar and turbulent flow.

$$R_e = \frac{\rho V \ell}{\mu}$$

The Reynolds number is a measure of the ratio of the inertia force on an element of fluid to the viscous force on an element. When these two types of forces are important in a given problem, the Reynolds number will play an important role. However, if the Reynolds number is very small this is an indication that the viscous forces are dominant in the problem. For large Reynolds number flows, viscous effects are small relative to inertial effects and for these cases it may be possible to neglect the effect of viscosity and consider the problem as one involving a “nonviscous” fluid.

The **Froude number** is distinguished from the other dimensionless group because that it contains the
acceleration of gravity, *g.*

$$F_r = \frac{V}{\sqrt{g\ell}}$$

The Froude number is a measure of the ratio of the inertia force on an element of fluid to the
weight of the element. It will generally be important in problems involving flows with free surfaces since gravity principally affects this type of flow. Is named in honor of William Froude (1810–

1879), a British civil engineer, mathematician, and naval architect who pioneered the use of
towing tanks for the study of ship design.

The **Euler number** can be interpreted as a measure of the ratio of pressure forces to inertial forces, where Δp is some characteristic pressure in the flow field. Very often the Euler number is written in terms
of a pressure difference, Δp so that $E_u = \frac{\Delta p}{\rho V^2}$. Also, this combination expressed as$\Delta p / \frac{1}{2}\rho V^2$
is called the pressure coefficient.

$$E_u = \frac{p}{\rho V^2}$$

The Euler number is named in honor of Leonhard Euler 11707–17832, a famous Swiss mathematician who pioneered work on the relationship between pressure and flow. For problems in which cavitation is of concern, the dimensionless group $(p_r - p_v)\ {}^1\!/_2\,\rho V^2$ is commonly used, where p_v is the vapor pressure and p_r is some reference pressure.

The **Cauchy number**

$$Ca = \frac{\rho v^2}{E_v}$$

and the **Mach number**

$$Ma = \frac{V}{c}$$

are important dimensionless groups in problems in which fluid compressibility is a significant
factor. Since the speed of sound, c, in a fluid is equal to $c = \sqrt{{}^{E_v}\!/_{\rho}}$ it follows that

$$Ma = V\sqrt{\frac{\rho}{E_v}}$$

and the square of the Mach number

$$Ma^2 = \frac{\rho V^2}{E_v} = Ca$$

is equal to the Cauchy number. Thus, either number 1but not both2 may be used in problems
in which fluid compressibility is important. When the Mach number is relatively small (say, less than 0.3), the inertial forces induced by the fluid motion are not sufficiently large to cause a significant change in the fluid density, and in this case the compressibility of the fluid can be neglected. The Mach number is the more commonly used parameter in compressible flow problems, particularly in the fields of gas dynamics and aerodynamics.

The **Strouhal number**

$$\mathrm{St} = \frac{\omega\ell}{\mathrm{V}}$$

is a dimensionless parameter that is likely to be important in unsteady, oscillating flow problems

in which the frequency of the oscillation is It represents a measure of the ratio of inertial forces due to the unsteadiness of the flow to the inertial forces due to changes in velocity from point to point in the flow field. This system of vortices, called a Kármán vortex trail named after Theodor von Kármán (1881–1963), a famous fluid mechanician, creates an oscillating flow at a discrete frequency, such that the Strouhal number can be closely correlated with the Reynolds number. Other types of oscillating flows is blood flow in arteries is periodic and can be analyzed by breaking up the periodic motion into a series of harmonic components, with each component having a frequency that is a multiple of the fundamental frequency,ω. Rather than use the Strouhal number in this type of problem, a dimensionless group formed by the product of St and Re is used; that is

$$\mathrm{St\ x\ Re} = \frac{\rho\omega\ell^2}{\mu}$$

The square root of this dimensionless group is often referred to as the frequency parameter.

The **Weber number**

$$We = \frac{\rho V^2 \ell}{\sigma}$$

may be important in problems in which there is an interface between two fluids. In this situation

the surface tension may play an important role in the phenomenon of interest. The Weber number can be thought of as an index of the inertial force to the surface tension force acting on a fluid element. The Weber number is named after Moritz Weber (1871–1951), a German professor of naval mechanics who was instrumental in formalizing the general use of common dimensionless groups as a basis for similitude studies.

7.7 Correlation of Experimental Data

The most important uses of dimensional analysis is as an aid in the efficient handling,

interpretation, and correlation of experimental data. The degree of difficulty involved in this process depends on the number of pi terms, and the nature of the experiments. The simplest problems are obviously those involving the fewest pi terms, and the following sections indicate how the complexity

of the analysis increases with the increasing number of pi terms.

7.7.1 Problems with One Pi Term

The pi theorem indicates that if the number of variables minus the number of reference dimensions is equal to unity, then only one pi term is required to describe the phenomenon. The functional relationship that must exist for one pi term is

$$\Pi_1 = C$$

Example: 7.3

Assume that the drag $\mathfrak{D}$, acting on a spherical particle that falls very slowly through a viscous fluid is a function of the particle diameter, *d*, the particle velocity, *V*, and the fluid viscosity, μ. Determine, with the aid of dimensional analysis, how the drag depends on the particle velocity.

$\mathfrak{D} = f(d, V, \mu)$

$\mathfrak{D} \doteq F, d \doteq L, V \doteq LT^{-1}, \mu \doteq FL^{-2}T$

Are four variables and three reference dimensions (F, L and T), according to the pi theorem, one pi term is required.

$$\prod_1 = \frac{\mathfrak{D}}{\mu V d} \qquad\qquad \frac{\mathfrak{D}}{\mu V d} = C$$

$$\mathfrak{D} = C\mu V d$$

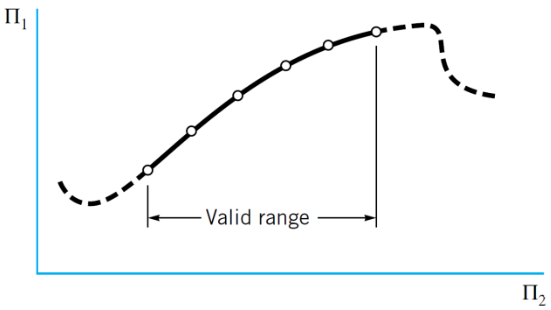
The drag varies directly with the velocity so that, $\boldsymbol{\mathfrak{D} \propto V}$.

7.7.2 Problems with Two Pi Terms

With two pi terms such that

$$\prod_1 = \emptyset(\prod_2)$$

the functional relationship among the variables can then be determined by varying $\prod_2$ and measuring the corresponding values of $\prod_1$. For this case the results can be conveniently presented in graphical form by plotting $\prod_1$ versus$\prod_2$ as is illustrated in Fig. 7.4



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Example 7.4

The relationship between the pressure drop per unit length along a smooth-walled, horizontal pipe and the variables that affect the pressure drop is to be determined experimentally. In the laboratory the pressure drop was measured over a 5-ft length of smooth-walled pipe having an inside diameter of 0.496 in. The fluid used was water at

$$60^{\circ}F \left(\begin{matrix} \mu = 2.34\ x\ 10^{-5}\ lb\ s/ft^2, \\ \rho = 1.94\ slugs/ft^3 \end{matrix} \right)$$

Tests were run in which the velocity was varied and the corresponding pressure drop measured. The results of these tests are shown below:

Velocity $\left(ft/_{s}\right)$	1.17	1.95	2.91	5.84	11.13	16.92	23.34	28.73
Pressure drop for 5-ft length $\left(lb/_{ft^2}\right)$	6.26	15.6	30.9	106	329	681	1200	1730

Make use of these data to obtain a general relationship between the pressure drops per unit
length and the other variables.

$$\Delta p = f(D, \rho, \mu, V)$$

Modeling and Similitude

A model is a representation of a physical system that may be used to predict the behavior of the system in some desired respect. Engineering model involves structures, aircrafts, ships, rivers and others. Also the prototype is the physical system for which the predictions are to be made. In many cases the model is smaller than the prototype. This is advantageous because when the model is larger than the prototype because it can be more easily to study. And with a successfully development of a valid model, it is possible to predict the behavior of the prototype under certain set of conditions.

Theory of models

This theory can be developed using the principles of dimensional analysis. For a given problem we can describe in terms of a set of pi terms, it is not needed to know the specific value of variable to perform the dimensional analysis. The pi terms, with no subscript are the ones that represent the prototype in the other hand the terms with the m subscript will be used to designate the model pi terms or variables.

$\Pi_{1m} = \Phi\ (\Pi_{2m}, \Pi_{3m,}\ldots\ldots\ldots\ldots\ldots\Pi_{nm})$

The prediction equation indicates that the measured value obtained from a model will be equal to the corresponding value for the prototype as long as the other pi terms are equal.

$\Pi_{2m}, = \Pi_2$

$\Pi_{3m} = \Pi_3$

$\Pi_{nm} = \Pi_n$

these equations provides, model design conditions or similarity requirements or modeling laws. As an example to determine the drag on a thin rectangular plate normal to a fluid with a velocity a dimensional analysis is performed. By establishing the pi theorem, the design of the model we obtain the equation 7.11. As we see to achieve similarity between the model and prototype behavior, all the corresponding pi terms must be equated between the model and prototype. Is important that when we equate pi terms involving lengths ratios they need to have a complete geometric similarity between the model and the prototype. Geometric scaling may extend the finest features of the system, such as surface roughness, or small protuberances on a structure, since these kinds of geometric features may significantly influence the flow. Also any deviation must be considered. This geometric scaling may be difficult to achieve, specially with the surface roughness because this one is very difficult to characterize and control.

The dynamic similarity is when the equality of these pi terms requires the ratio of like forces to be the same in model and the prototype. Thus when we have both the geometric and dynamic similarity the streamline patterns are the same and also the corresponding velocity ratios and acceleration ratios are constant throughout the flow field is called the kinematic similarity. To have a completely similarity between model and prototype, we must maintain geometric, dynamic and kinematic similarity between the two systems.

Model scales

If in a given problem there are two variables the resulting similarity requirement based on a pi term it is define as the length scale. For true models there will be only one length scale. Also there are other scales such as the velocity scale, density scale and viscosity scale. The scale can be defined for any variable in the problem.

l_m/l Length Scale, $\mathbf{V_m/V}$ Velocity Scale, ρ_m/ρ Density Scale, μ_m/μ Viscosity Scale

Practical Aspects of using models

Validation of model design, the purpose of the model is to predict the effects of certain proposed changes in a given prototype, and in this instance some actual prototype data may be available. The model can be designated, constructed and tested and the model prediction can be compared with these data. If the agreement is satisfactory, the model can be changed in the desired manner and the effect on the prototype can be predicted with increased confidence. Another useful is to run test with a series of different sizes. A necessary condition for the validity of the model design is that an accurate prediction de made between any pair of models, since one can always be considered as a model of the other. If the agreement between models cannot be achieve in these tests, there is no reason to expect that the same model design can be used to predict prototype behavior correctly.

Distorted models are the models for which one or more of the similarity requirements are not satisfied. An example of distorted models occurs in the study of open channel or free-surface flows, usually the Reynolds numbers and Froude number are involved.

Distorted models can arise for a variety of reasons. The classic example of a distorted model occurs in the study of open channel or free-surface flows. In these problems both the Reynolds number, and the Froude number, are involved.

$\# Reynolds = \frac{\rho V \ell}{\mu}$ $\qquad$ $\# Froude = \frac{V}{g\ell}$

Froude number similarity requires $\frac{V_m}{\sqrt{g_m \ell_m}} = \frac{V}{\sqrt{g\ell}}$. To same gravitational field, the required velocity scale is

$\frac{V_m}{V} = \sqrt{\frac{\ell_m}{\ell}} = \sqrt{\lambda_\ell}$.

Reynolds number similarity requires $\frac{\rho_m V_m \ell_m}{\mu_m} = \frac{\rho V \ell}{\mu}$ and the velocity scale is $\frac{V_m}{V} = \frac{\mu_m \rho \ell}{\mu \rho_m \ell_m}$.

Since the velocity scale must be equal to the square root of the length scale, it follows that $\frac{V_m}{V} = (\lambda_\ell)^{3}/_{2}$.

Where the kinematic viscosity is equal $V = {}^{\mu}/_{\rho}$.

Although in principle it may be possible to satisfy this design condition, it may be quite difficult, to find a suitable model fluid, particularly for small length scales.

The interpretation of results obtained with the distorted model is more difficult than the interpretation of results obtained with *true models* for which all similarity requirements are met. The success of using distorted models depends to a large extent on the skill and experience of the investigator responsible for the design of the model and in the interpretation of experimental data obtained from the model.

7.9 Some Typical Model Studies

7.9.1. Flow Through Closed Conduits

Common examples of this type of flow include pipe flow and flow through valves, fittings, and metering devices. Although the conduits are often circular, they could have other shapes as well and may contain expansions or contractions. Since there are no fluid interfaces or free surfaces, the dominant forces are inertial and viscous so that the Reynolds number is an important similarity parameter. For low Mach numbers (Ma<0.3), compressibility effects are usually negligible for both the flow of liquids or gases. Generally the geometric characteristics can be described by a series of length terms, $\ell_1, \ell_2, \ell_3, \dots, \ell_i$, and ℓ, where ℓ is some particular length dimension for the system. Such a series of length terms leads to

a set of pi terms of the form $\Pi_i = \frac{\ell_i}{i}$, where $i = 1,2, \dots$ and so on.

If the average height of surface roughness elements is defined as E, then the pi term representing roughness will be $\frac{E}{\ell}$. This parameter indicates that for complete geometric similarity, surface roughness would also have to be scaled. Note that this implies that for length scales less than 1, the model surfaces should be smoother than those in the prototype since $E_m = \lambda_\ell E$.

For flow in closed conduits at low Mach numbers, any dependent pi term can be expressed as $Dependent\ pi\ term = \boldsymbol{\Phi}(\frac{\boldsymbol{\ell_i}}{\boldsymbol{\ell}}, \frac{\boldsymbol{E}}{\boldsymbol{\ell}}, \frac{\boldsymbol{\rho V \ell}}{\boldsymbol{\mu}})$**.** This is a general formulation for this type of problem. The first two pi terms of the right side of the equation lead to the requirement of geometric similarity so that $\frac{\ell_{im}}{\ell_i} = \frac{E_m}{E} = \frac{\ell_m}{\ell} = \lambda_\ell$. This result indicates that the investigator is free to choose a length scale, λ_ℓ, but once this
scale is selected, all other pertinent lengths must be scaled in the same ratio.

The velocity scale is established so that $\frac{V_m}{V} = \frac{\mu_m}{\mu}\frac{\rho}{\rho_m}\frac{\ell}{\ell_m}$ and the actual value of the velocity scale depends on the viscosity and density scales, as well as the length scale. If the same fluid is used (with $\mu_m = \mu$ and $\rho_m = \rho$), then $\frac{V_m}{V} = \frac{\ell}{\ell_m}$. Thus, $V_m = \frac{V}{\lambda_\ell}$, which indicates that the fluid velocity in the model will be larger than that in the prototype for any length scale less than 1.

Example 7.6

Model tests are to be performed to study the flow through a large valve having a 2-ft-diameter inlet and carrying water at a flowrate of 30 cfs. The working fluid in the model is water at the same temperature as that in the prototype. Complete geometric similarity exists between model and prototype, and the model inlet diameter is 3 in. Determine the required flowrate in the model.

Solution:

To ensure dynamic similarity, the model tests should be run so that $\frac{V_m D_m}{v_m} = \frac{VD}{v}$ where *V* and *D* correspond to the inlet velocity and diameter, respectively. Since the same fluid is to be used in model and prototype, $v = v_m$, and therefore $\frac{V_m}{V} = \frac{D}{D_m}$. The discharge, *Q*, is equal to *VA*, where *A* is the inlet area, so

$$\frac{Q_m}{Q} = \frac{V_m A_m}{VA} = \left(\frac{D}{D_m}\right)\left(\frac{((^{\Pi}/_{4})D_m^2)}{((^{\Pi}/_{4})D^2)}\right) = \frac{D_m}{D} = \frac{^3/_{12}ft}{2ft} = {}^1/_8$$

$$Q_m = ({}^1/_8) \times 30\,{}^{ft^3}/_s = 3.75cfs$$

Two additional points should be made with regard to modeling flows in closed conduits. First, for large Reynolds numbers, inertial forces are much larger than viscous forces, and in this case it may be possible to neglect viscous effects. The important practical consequence of this is that it would not be necessary to maintain Reynolds number similarity between model and prototype. The second point relates to the possibility of cavitation in flow through closed conduits. The use of models to study cavitation is complicated, since it is not fully understood how vapor bubbles form and grow. The initiation of bubbles seems to be influenced by the microscopic particles that exist in most liquids, and how this aspect of the problem influences model studies is not clear.

7.9.2 Flow Around Immersed Bodies

Models have been widely used to study the flow characteristics associated with bodies that are completely immersed in a moving fluid. Examples include flow around aircraft, automobiles, golf balls, and buildings. Modeling laws for these problems is geometric and Reynolds number similarity is required. Since there are no fluid interfaces, surface tension (and therefore the Weber number) is not important. Also, gravity will not affect the flow patterns, so the Froude number need not be considered. The Mach number will be important for high-speed flows in which compressibility becomes an important factor, but for incompressible fluids 1such as liquids or for gases at relatively low speeds2 the Mach number can be omitted as a similarity requirement. In this case, a general formulation for these problems is $Dependent\,pi\ term = \boldsymbol{\Phi}(\frac{\boldsymbol{\ell_i}}{\boldsymbol{\ell}}, \frac{\boldsymbol{E}}{\boldsymbol{\ell}}, \frac{\boldsymbol{\rho V \ell}}{\boldsymbol{\mu}})$**,** where $\boldsymbol{\ell}$ is some characteristic length of the system and $\boldsymbol{\ell_i}$ represents other pertinent lengths, $\frac{\boldsymbol{E}}{\boldsymbol{\ell}}$ is the relative roughness of the surface (or surfaces), and $\frac{\boldsymbol{\rho V \ell}}{\boldsymbol{\mu}}$ is the Reynolds number.

Frequently, the dependent variable of interest for this type of problem is the drag, G, developed on the body, and in this situation the dependent pi term would usually be expressed in the form of a *drag coefficient*, C_D, where $C_D = \frac{G}{1/2 \rho V^2 \ell^2}$. The numerical factor, $1/2$, is arbitrary but commonly included, and ℓ^2 is usually taken as some representative area of the object. Thus, drag studies can be undertaken with the formulation

$$\frac{G}{1/2 \rho V^2 \ell^2} = C_D = \boldsymbol{\Phi}(\frac{\boldsymbol{\ell_i}}{\boldsymbol{\ell}}, \frac{\boldsymbol{E}}{\boldsymbol{\ell}}, \frac{\boldsymbol{\rho V \ell}}{\boldsymbol{\mu}}),$$

Then $\frac{G}{1/2 \rho V^2 \ell^2} = \frac{G_m}{1/2 \rho_m V_m^2 \ell_m^2}$.

Then $V_m = \frac{\ell}{\ell_m} V$, therefore, the required model velocity will be higher than the prototype velocity for $\frac{\ell}{\ell_m}$ greater than 1. Since this ratio is often relatively large, the required value of V_m may be large. For example, for a $\frac{1}{10}$-length scale, and a prototype velocity of 50 mph, the required model velocity is 500 mph. This is a value that is unreasonably high to achieve with liquids, and for gas flows this would be in the range where compressibility would be important in the model (but not in the prototype).

Example 7.7

The drag on an airplane cruising at 240 mph in standard air is to be determined from tests on a 1 : 10 scale model placed in a pressurized wind tunnel. To minimize compressibility effects, the air speed in the wind tunnel is also to be 240 mph. Determine the required air pressure in the tunnel (assuming the same air temperature for model and prototype), and the drag on the prototype corresponding to a measured force of 1 lb on the model.

Solution:
For this example, $V_m = V$ and ${}^{\ell_m}/_{\ell} = 1/10$ so that

$$\frac{\rho_m}{\rho} = \frac{\mu_m}{\mu}\frac{V}{V_m}\frac{\ell}{\ell_m} = \frac{\mu_m}{\mu}(1)(10) = 10\frac{\mu_m}{\mu}$$

This result shows that the same fluid with $\rho_m = \rho$ and $\mu_m = \mu$ cannot be used if Reynolds number similarity is to be maintained. One possibility is to pressurize the wind tunnel to increase the density of the air. We assume that an increase in pressure does not significantly change the viscosity so that the required increase in density is given by the relationship $\frac{\rho_m}{\rho} = 10$. For an ideal gas, $P = \rho RT$ so that $\frac{P_m}{P} = \frac{\rho_m}{\rho}$, for constant temperature. Therefore, the wind tunnel would need to be pressurized so that $\frac{P_m}{P} = 10$. Since the prototype operates at standard atmospheric pressure, the required pressure in the wind tunnel is 10 atmospheres.

Thus, we see that a high pressure would be required and this could not be easily or inexpensively achieved. However, under these conditions Reynolds similarity would be attained and the drag could be obtained from the equation $\frac{G}{{}^{1}/_{2}\rho V^2 \ell^2} = \frac{G_m}{{}^{1}/_{2}\rho_m V_m^2 \ell_m^2}$ or $G = \frac{\rho}{\rho_m}(\frac{V}{V_m})^2(\frac{\ell}{\ell_m})^2 G_m = \left(\frac{1}{10}\right)(1)^2(10)^2 G_m = 10G_m$. Thus, for a drag of 1 lb on the model the corresponding drag on the prototype is $G = 10lb$.